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Reset Adaptive Observer for a Class of Nonlinear Systems

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***Abstract*—This paper proposes a novel kind of state estimator called reset adaptive observer (ReO). A ReO is an adaptive observer consisting of an integrator and a reset law that resets the output of the integrator depending on a predefined condition. The main contribution of this paper is that the reset element theory is applied for the first time to the nonlinear adaptive observer framework. The introduction of the reset element in the adaptive law can decrease the overshooting and settling time of the estimation process without sacrificing the rising time. The stability and convergence LMI-based analysis of the proposed ReO is addressed and, additionally, an easily computable method to determine the $\mathcal{L}_2$ gain of the ReO dealing with noise-corrupted systems is presented.**

***Index Terms*—Estimation, Reset Control, Output feedback and Observers, Hybrid systems.**

I. Introduction

An adaptive observer is a recursive algorithm for joint state and parameter estimation in dynamic systems. This kind of algorithm plays a key role in many applications such as failure detection, monitoring, and fault tolerant control. The research on adaptive observers started in the 1970s. Initially, it was focused on linear time invariant systems [1], and afterwards on nonlinear systems [2], [3], [4]. All those works were characterized by having only a proportional feedback term of the output observation error in both state observer and parameter adaptation law. This proportional approach ensures a bounded estimation of the state and the unknown parameter, assuming a persistent excitation condition as well as the lack of disturbances. The performance of proportional adaptive observers was improved by adding an integral term to the adaptive laws, [5], [6], [7]. This additional term can increase the steady state accuracy and improve the robustness against modeling errors and disturbances.

However, since the adaptive laws are still linear, they have the inherent limitations of linear feedback control. Namely, they cannot decrease the settling time and the overshoot of the estimation process simultaneously. Therefore, a trade-off between both requirements is needed. Nevertheless, this limitation can be solved by adding a reset element. A reset element consists of an integrator and a reset law that resets the output of the integrator as long as the reset condition holds. Reset elements were introduced by Clegg in 1958 [8], who proposed an integrator which was reset to zero when its input is zero. In 1974, Horowitz generalized that initial work substituting the Clegg integrator by a more general structure called the first order reset element (FORE), [9]. During the last years, the research on the stability analysis and switching stabilization for reset systems is attracting the attention of many academics and engineers. The main difference between the state-of-art reset control works is how to address the stability analysis. Although some authors have recently included the reset time intervals in the stability analysis [10], [11], the reset time independent approach is still the most popular. A general analysis for such time independent reset control systems can be found in [12]. There, the authors modified the reset condition in such a manner that the system is reset when its input and output have different sign, rather than as long as its input is equal to zero. This is the main difference of [12], compared with other relevant time independent reset control works [13], [14].

Although the research on reset elements is still an open and challenging topic, this research has been mainly focused on control issues. The first application of the reset elements to the adaptive observer framework is [15]. There, the authors proposed a new sort of adaptive observer called reset adaptive observer (ReO). A ReO is an adaptive observer whose integral term has been substituted for a reset element. The reset condition of the ReO is based on the approach proposed by [12]. Since the integral term is reset as long as the output estimation error and the integrated estimation error have opposite sign, the reset time intervals are unknown a priori. The introduction of the reset element in the adaptive laws can improve the performance of the observer, as it is possible to decrease the overshoot and settling time of the estimation process simultaneously.

This paper extends the previous version [15], which now considers nonlinear formulation as well as joint state and uncertain parameter estimation. In Section II, the ReO formulation for a class of nonlinear systems is presented. In Section III, a LMI-based stability condition which guarantees the convergence and stability of the estimation process is developed. Besides, an easily computable method to obtain the $\mathcal{L}_2$ gain of the ReO dealing with noise-corrupted system is presented. Simulation results are presented in Section IV to test the performance of our proposed ReO compared with traditional PIAO. Finally, concluding remarks are given in Section V.

Notation: In the following, we use the notation $(x, y) = \left[x^{\mathrm{T}}\ y^{\mathrm{T}}\right]^{\mathrm{T}}$. Given a state variable x of a hybrid system with switches, we will denote its time derivative with respect to the time by $\dot{x}$, and we will denote the value of the state variable after the switch by x^+. Note that we omit its time argument and we write $x(t)$ as x. Additionally, an input signal v is persistently exciting if there exists three positive reals k_1, k_2, T_p such that $k_1 \leq \int_t^{t+T_p} v^{\mathrm{T}}(\tau)v(\tau)d\tau \leq k_2$ for all $t \geq 0$.

II. Reset Adaptive Observer Formulation

In this paper we address the problem of joint state and unknown parameter estimation for uncertain nonlinear systems which can be described by

$$\begin{aligned} \dot{x} &= Ax + Bu + \Delta\phi\theta + B_w w \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $\theta \in \mathbb{R}^p$ is the unknown constant parameter vector which can be used to represent modeling uncertainties, $w \in \mathbb{R}^w$ is the disturbance vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $\Delta \in \mathbb{R}^{n\times 1}$, and $B_w \in \mathbb{R}^{n\times w}$ are known constant matrices. The nonlinearity $\phi \in \mathbb{R}^{1\times p}$ is a time-varying matrix which depends on the input u and/or the output y. In addition, u and ϕ are assumed persistently exciting, and the pair (A, C) is assumed observable. As it is shown in [2], a class of nonlinear systems can be formulated as the system described by (1) through a change of coordinates.

The ReO dynamics are described as follows:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + \Delta\phi\hat{\theta} + K_I\zeta + K_P\tilde{y} \\ \dot{\hat{\theta}} &= \Gamma\phi^{\mathrm{T}}\tilde{y} - \Gamma\sigma_s\hat{\theta} \\ \hat{y} &= C\hat{x} \end{aligned} \tag{2}$$

where $\hat{x}$ is the estimated state, K_I and K_P represent the integral and proportional gain respectively, $\tilde{y} = C\tilde{x} = C(x - \hat{x})$ is the output estimation error, $\Gamma \in \mathbb{R}^{p\times p}$ is a positive definite matrix, and σ_s is a switching leakage term defined as [16]

$$\sigma_s = \left\{ \begin{array}{ll} 0 & \text{if } \|\hat{\theta}\| < T_h \\ \sigma & \text{if } \|\hat{\theta}\| \geq T_h \end{array} \right\}, \tag{3}$$

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which guarantees that the estimated parameter $\hat{\theta}$ is bounded and remains within a predefined threshold T_h, which is chosen to be large enough in such a manner that $T_h > \|\theta\|$ based on prior knowledge of the system, and where σ is a small positive constant. Although the discontinuity of σ_s may cause small oscillations on the switching surface $\|\hat{\theta}\| = T_h$, these oscillations never appear working on nominal conditions. They only appear after a failure when the parameter estimate is drifting to infinity and it reaches the switching surface. This is an effective and simple method for eliminating parameter drift and keeping the parameter estimate bounded.

In addition, ζ is the reset integral term which can be computed as

$$\left.\begin{array}{rcl} \dot{\zeta} & = & A_\zeta \zeta + B_\zeta \tilde{y} \\ \dot{\tau} & = & 1 \end{array}\right\} \text{ if } (\tilde{y},\zeta) \in \mathcal{F} \text{ or } \tau \leq \rho, \qquad \left.\begin{array}{rcl} \zeta^+ & = & A_r \zeta \\ \tau^+ & = & 0 \end{array}\right\} \text{ if } (\tilde{y},\zeta) \in \mathcal{J} \text{ and } \tau \geq \rho, \tag{4}$$

where $A_\zeta \in \mathbb{R}$ and $B_\zeta \in \mathbb{R}$ are two tuning scalars which regulate the transient response of ζ, and A_r is the reset matrix. Specifically, we define $A_r = 0$, since the reset integral term ζ is reset to zero when $\tilde{y} \cdot \zeta \leq 0$. To avoid Zeno solutions, the reset term dynamics (4) has been modified imposing temporal regularization. We use the notation proposed by [12], based on including an auxiliary variable τ which guarantees that the time interval between any two consecutive resets is not smaller than ρ which is a small enough positive number.

As it was shown in [15], the ReO can be regarded as a hybrid system with a *flow set* $\mathcal{F}$ and a *jump or reset set* $\mathcal{J}$. On one hand, when $(\tilde{y},\zeta) \in \mathcal{F}$, that is, if $\tilde{y}$ and ζ have the same sign, the ReO behaves as a proportional integral observer. On the other hand, if the pair $(\tilde{y},\zeta) \in \mathcal{J}$, that is, if $\tilde{y}$ and ζ have different sign, the integral term is reset according to the reset map A_r. According to these statements and since $\tilde{y} = C\tilde{x}$, the definition of both sets can be formalized by using the following augmented representation:

$$\mathcal{F} := \left\{\eta : \eta^{\mathrm{T}} M \eta \geq 0\right\}, \quad \mathcal{J} := \left\{\eta : \eta^{\mathrm{T}} M \eta \leq 0\right\}, \tag{5}$$

where $\eta = [\tilde{x} \;\; \zeta]^{\mathrm{T}}$ and $M = M^{\mathrm{T}} = \begin{bmatrix} 0 & C^{\mathrm{T}} \\ C & 0 \end{bmatrix}$.

III. Stability and Convergence Analysis

In this section we analyze the stability and convergence of the ReO defined by (2) and (4) applied to nonlinear systems described by (1). Firstly, the error dynamics which are involved in the estimation process are shown. Secondly, a computable method to determine the stability of the ReO assuming absence of disturbances is given. Thirdly, stability results are extended to noise-corrupted systems. For this reason, a method to compute the $\mathcal{L}_2$ gain minimizing the effect of the disturbances on the output estimation error is also provided.

A. Observer error dynamics

Let us begin analyzing the error system dynamics which can be obtained subtracting (2) from (1). Then, the state error dynamics $\tilde{x} = x - \hat{x}$ is defined by:

$$\dot{\tilde{x}} = (A - K_P C)\tilde{x} + \Delta\phi\tilde{\theta} - K_I \zeta + B_w w \tag{6}$$

while the parameter error dynamics $\tilde{\theta} = \theta - \hat{\theta}$ is described by:

$$\dot{\tilde{\theta}} = -\Gamma\phi^{\mathrm{T}}\tilde{y} + \Gamma\sigma_s\hat{\theta} \tag{7}$$

The state error dynamics can be augmented by connecting (6) to (4) as follows:

$$\left.\begin{array}{rcl} \dot{\eta} & = & A_\eta\,\eta + B_\Delta\,\phi\tilde{\theta} + B_\eta\,w \\ \dot{\tau} & = & 1 \\ \xi & = & C_\eta \eta \end{array}\right\} \text{ if } \eta \in \mathcal{F} \text{ or } \tau \leq \rho, \qquad \left.\begin{array}{rcl} \eta^+ & = & A_R\,\eta \\ \tau^+ & = & 0 \\ \xi^+ & = & \xi \end{array}\right\} \text{ if } \eta \in \mathcal{J} \text{ and } \tau \geq \rho, \tag{8}$$

where

$$A_\eta = \begin{bmatrix} A - K_P C & -K_I \\ B_\zeta C & A_\zeta \end{bmatrix}, A_R = \begin{bmatrix} I & 0 \\ 0 & A_r \end{bmatrix}, \quad B_\Delta = \begin{bmatrix} \Delta \\ 0 \end{bmatrix}, B_\eta = \begin{bmatrix} B_w \\ 0 \end{bmatrix}, C_\eta = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{9}$$

Notice that the parameter error dynamics does not change after resets, that is, $\tilde{\theta}^+ = \tilde{\theta}$. Only the reset term ζ of the augmented state error dynamics η is modified through A_R after resets, since $\eta^+ = A_R\,\eta$. It is also worth pointing out that the output of the augmented error dynamics (8) is equal to the output of the ReO observer (2), that is, $\tilde{y} = C\tilde{x} = C_\eta\,\eta = \xi$.

Under normal conditions, the ReO flows if $\eta \in \mathcal{F}$ and is reset when $\eta \in \mathcal{J}$. However, due to the effect of the temporal regularization, if the ReO hits the reset surface and $\tau \leq \rho$, it has to keep flowing until $\tau \geq \rho$. Meanwhile, the state η might overflow into the adjacent reset region. If so, the ReO has actually been flowing in the slightly inflated flow region $\mathcal{F}_\epsilon$ which considers the original flow region $\mathcal{F}$ and a slight portion of the jump set adjacent to the original flow set boundary. It is formally defined as $\mathcal{F}_\epsilon := \{\eta : \eta^{\mathrm{T}} M \eta + \epsilon\eta^{\mathrm{T}}\eta\}$, where $\epsilon(\rho) \geq 0$ represents how the set is inflated [12]. Since $\epsilon \to 0$ as $\rho \to 0$, an arbitrarily small ρ can be chosen so that the effect of ϵ is small enough to be neglected [17]. For stability purposes, we have to prove that $\dot{V}(\eta,\tilde{\theta})$ is negative in any region wherein the state η can flow, and that $V(\eta^+,\tilde{\theta}^+) \leq V(\eta,\tilde{\theta})$ for any region wherein the state η is reset. In particular, since $\mathcal{F} \subseteq \mathcal{F}_\epsilon$, we have to check $\dot{V}(\eta,\tilde{\theta}) < 0$ for all $\eta \in \mathcal{F}_\epsilon$, and $V(\eta^+,\tilde{\theta}^+) \leq V(\eta,\tilde{\theta})$ for all $\eta \in \mathcal{J}$.

Given that $\mathcal{F}_\epsilon$ slightly overflows into the jump set $\mathcal{J}$, the following assumption is needed to guarantee that the solution will be mapped to the flow set $\mathcal{F}$ after each reset and, consequently, there are no trajectories that keep flowing and jumping within $\mathcal{J}$.

Assumption 1. *The reset observer described by (2)-(4) is such that* $\eta \in \mathcal{J} \;\Rightarrow\; A_R\,\eta \in \mathcal{F}$.

It is important to note that this assumption is quite natural to assume for hybrid systems [18], and consequently, this condition is commonly used in most of current reset system formulations available in literature [12], [17], [19].

B. State stability analysis

Now, taking the augmented error dynamics (7) and (8) into account, we state a sufficient condition to prove the quadratical stability of our proposed ReO assuming absence of disturbances, that is, $w = 0$. This analysis is based on a LMI approach.

Theorem 1. *For given A_η, B_η, C_η, B_Δ, A_R and ϵ, the augmented error dynamics shown in (7) and (8) with $w = 0$ is quadratically stable, if there exist a matrix $P = P^{\mathrm{T}} > 0$ and scalars $\tau_F \geq 0$ and $\tau_J \geq 0$ subject to*

$$\begin{array}{rcl} A_\eta^{\mathrm{T}} P + P A_\eta + \tau_F(M + \epsilon I) & < & 0, \\ A_R^{\mathrm{T}} P A_R - P - \tau_J M & \leq & 0, \\ P B_\Delta & = & C_\eta^{\mathrm{T}}. \end{array} \tag{10}$$

Proof: Let us begin considering the following quadratic Lyapunov function for the error dynamics described by (7) and (8):

$$V(\eta,\tilde{\theta}) = \eta^{\mathrm{T}}\, P\,\eta + \tilde{\theta}^{\mathrm{T}}\,\Gamma^{-1}\,\tilde{\theta} \tag{11}$$

where $P = P^{\mathrm{T}} > 0$ and $\Gamma = \Gamma^{\mathrm{T}} > 0$.

To prove the quadratically stability of our proposed ReO, we have to check that:

$$\begin{array}{ll} \dot{V}(\eta,\tilde{\theta}) < 0 & \eta \in \mathcal{F}_\epsilon \\ V(\eta^+,\tilde{\theta}^+) \leq V(\eta,\tilde{\theta}) & \eta \in \mathcal{J} \end{array} \tag{12}$$

Since $\mathcal{F}_\epsilon := \left\{\eta : \eta^{\mathrm{T}}\, M\,\eta + \epsilon\eta^{\mathrm{T}}\eta \geq 0\right\}$, employing the S-procedure [20], the first term of (12) is equivalent to the existence of $\tau_F \geq 0$ such that

$$\dot{V}(\eta,\tilde{\theta}) < -\tau_F\eta^{\mathrm{T}}(M+\epsilon I)\eta \tag{13}$$

Then, let us take derivative of (11) to obtain

$$\begin{array}{rcl} \dot{V}(\eta,\tilde{\theta}) & = & \dot{\eta}^{\mathrm{T}}P\eta + \eta^{\mathrm{T}}P\dot{\eta} + \dot{\tilde{\theta}}^{\mathrm{T}}\,\Gamma^{-1}\,\tilde{\theta} + \tilde{\theta}^{\mathrm{T}}\,\Gamma^{-1}\,\dot{\tilde{\theta}} \\ & = & \eta^{\mathrm{T}}(A_\eta^{\mathrm{T}}P + PA_\eta)\eta + \eta^{\mathrm{T}}\left(PB_\Delta\phi - C_\eta^{\mathrm{T}}\phi\right)\tilde{\theta} \\ & + & \tilde{\theta}^{\mathrm{T}}\left(\phi^{\mathrm{T}}B_\Delta^{\mathrm{T}}P - \phi^{\mathrm{T}}C_\eta\right)\eta + \varphi \end{array} \tag{14}$$

where $\varphi = 2\sigma_s\hat{\theta}^{\mathrm{T}}\tilde{\theta}$. Notice that σ_s should be designed in such a manner that φ becomes non-positive in the space of the parameter estimates. Thus, let us prove that φ has an upper non-positive bound by using the Cauchy-Schwarz inequality and $T_h > \|\theta\|$

$$\begin{array}{rcl} \varphi & = & 2\sigma_s\hat{\theta}^{\mathrm{T}}\tilde{\theta} = 2\sigma_s\left(\hat{\theta}^{\mathrm{T}}\theta - \hat{\theta}^{\mathrm{T}}\hat{\theta}\right) \\ & \leq & 2\sigma_s\left(\|\hat{\theta}\|\|\theta\| - \|\hat{\theta}\|\|\hat{\theta}\|\right) < 2\sigma_s\left(\|\hat{\theta}\|\left(T_h - \|\hat{\theta}\|\right)\right) \end{array} \tag{15}$$

According to the first term of (3), if $\|\hat{\theta}\| < T_h$, $\varphi = 0$ since $\sigma_s = 0$. On the other hand, when $\|\hat{\theta}\| \geq T_h$, $\varphi < 0$ since $T_h - \|\hat{\theta}\| \leq 0$. Consequently, $\varphi \leq 0$ is proved.

Rearranging terms of equations (13) and (14), and by using $PB_\Delta = C_\eta^{\mathrm{T}}$, the first term of (12) holds if the following inequality is satisfied

$$\begin{array}{l} \eta^{\mathrm{T}}(A_\eta^{\mathrm{T}}P + PA_\eta)\eta + \tau_F\eta^{\mathrm{T}}(M+\epsilon I)\eta + \varphi \\ \leq \eta^{\mathrm{T}}(A_\eta^{\mathrm{T}}P + PA_\eta)\eta + \tau_F\eta^{\mathrm{T}}(M+\epsilon I)\eta < 0, \end{array} \tag{16}$$

that can be rearranged as an equivalent LMI problem in the variables $P > 0$ and $\tau_F \geq 0$

$$A_\eta^{\mathrm{T}}P + PA_\eta + \tau_F(M+\epsilon I) < 0, \tag{17}$$

which is the first term of (10) and consequently, proves the first equation of (12). Similarly, employing again the S-procedure, the second term of (12) holds if there exits $\tau_J \geq 0$ such that

$$V(\eta^+,\tilde{\theta}^+) \leq V(\eta,\tilde{\theta}) + \tau_J\eta^{\mathrm{T}}M\eta, \tag{18}$$

which is equivalent to

$$\eta^{\mathrm{T}}A_R^{\mathrm{T}}PA_R\eta - \eta^{\mathrm{T}}P\eta - \tau_J\eta^{\mathrm{T}}M\eta \leq 0. \tag{19}$$

Rearranging terms, (18) can be also rewritten as an equivalent LMI problem in the variables $P > 0$ and $\tau_J \geq 0$ as follows

$$A_R^{\mathrm{T}}PA_R - P - \tau_J M \leq 0, \tag{20}$$

which is the second term of (10) and proves the second equation of (12) and, as a consequence, completes the proof of the theorem.

Since the Lyapunov function candidate (11) relies on the the augmented state error η and the parameter error $\tilde{\theta}$, we can prove the asymptotic convergence of η and $\tilde{\theta}$ and the boundedness of all the signals involved in the estimation process by satisfying the conditions shown in Theorem 1. $\tilde{\theta}$ does not appear explicitly on the resultant LMIs, since the matching condition $PB_\Delta = C_\eta^{\mathrm{T}}$ and the fact that φ has an upper non-positive bound allows us to remove some terms from the Lyapunov function. ∎

C. Input-output stability analysis

Now, we present our results on the input-output properties of the ReO. The aim is to develop a ReO such as the effect of the disturbance w on the output estimation error ξ is minimized. For this reason, let us define the $\mathcal{L}_2$ gain of the system (8) as $\mathcal{L}_2 = \sup_{\|w\|_2\neq 0}\frac{\|\xi\|_2}{\|w\|_2}$, where the $\mathcal{L}_2$ norm $\|u\|_2^2$ of a signal u is defined $\|u\|_2^2 = \int_0^\infty u^{\mathrm{T}}u\,dt$, and sup means *supremum* which is taken over all non-zero trajectories of (8).

Additionally, the following lemma that will be used in the sequel is enunciated [20],

Lemma 1. *The $\mathcal{L}_2$ gain of a LTI system with an input signal u and an output signal y is less than γ, if there exists a quadratic function $V(x) = x^{\mathrm{T}}Qx$, $Q = Q^{\mathrm{T}} > 0$ and $\gamma > 0$ such that*

$$\dot{V}(x) < \gamma^2u^{\mathrm{T}}u - y^{\mathrm{T}}y \tag{21}$$

Now, we apply this lemma to the augmented error dynamics (7) and (8) to obtain the following theorem.

Theorem 2. *For given A_η, B_η, C_η, B_Δ, A_R, and ϵ, the augmented error dynamics shown in (7) and (8) is quadratically stable and have a $\mathcal{L}_2$ gain from w to ξ which is smaller than γ, if there exist a matrix $P = P^{\mathrm{T}} > 0$ and scalars $\tau_F \geq 0$, $\tau_J \geq 0$ and $\gamma > 0$ subject to*

$$\begin{array}{rcl} \begin{bmatrix} A_\eta^{\mathrm{T}}P + PA_\eta + C_\eta^{\mathrm{T}}C_\eta + \tau_F(M+\epsilon I) & PB_\eta \\ B_\eta^{\mathrm{T}}P & -\gamma^2 I \end{bmatrix} & < & 0, \\ A_R^{\mathrm{T}}PA_R - P - \tau_J M & \leq & 0, \\ PB_\Delta & = & C_\eta^{\mathrm{T}}. \end{array} \tag{22}$$

Proof: To prove the stability of our proposed ReO and that the $\mathcal{L}_2$ gain from w to ξ is smaller than γ, we have to check that:

$$\begin{array}{ll} \dot{V}(\eta,\tilde{\theta}) < \gamma^2w^{\mathrm{T}}w - \xi^{\mathrm{T}}\xi & \eta \in \mathcal{F}_\epsilon \\ V(\eta^+,\tilde{\theta}^+) \leq V(\eta,\tilde{\theta}) & \eta \in \mathcal{J} \end{array} \tag{23}$$

The first equation of (23) relies on (21) and the second equation of (23) is equal to the second equation of (12) which has been already proved. Then, let us concentrate on the first equation of (23). Again, since $\mathcal{F}_\epsilon := \left\{\eta : \eta^{\mathrm{T}}\, M\,\eta + \epsilon\eta^{\mathrm{T}}\eta \geq 0\right\}$ and employing the S-procedure, the first term of (23) is equivalent to the existence of $\tau_F \geq 0$ such that

$$\dot{V}(\eta,\tilde{\theta}) < \gamma^2w^{\mathrm{T}}w - \xi^{\mathrm{T}}\xi - \tau_F\eta^{\mathrm{T}}(M+\epsilon I)\eta \tag{24}$$

In this case, the time derivative of (11) is

$$\begin{array}{rcl} \dot{V}(\eta,\tilde{\theta}) & = & \dot{\eta}^{\mathrm{T}}P\eta + \eta^{\mathrm{T}}P\dot{\eta} + \dot{\tilde{\theta}}^{\mathrm{T}}\,\Gamma^{-1}\,\tilde{\theta} + \tilde{\theta}^{\mathrm{T}}\,\Gamma^{-1}\,\dot{\tilde{\theta}} \\ & = & \eta^{\mathrm{T}}(A_\eta^{\mathrm{T}}P + PA_\eta)\eta + w^{\mathrm{T}}B_\eta^{\mathrm{T}}P\eta + \eta^{\mathrm{T}}PB_\eta w \\ & + & \eta^{\mathrm{T}}PB_\Delta\phi\tilde{\theta} + \tilde{\theta}^{\mathrm{T}}\phi^{\mathrm{T}}B_\Delta^{\mathrm{T}}P\eta \\ & - & \eta^{\mathrm{T}}C_\eta^{\mathrm{T}}\phi\tilde{\theta} - \tilde{\theta}^{\mathrm{T}}\phi^{\mathrm{T}}C_\eta\eta + \varphi \end{array} \tag{25}$$

Rearranging terms of equations (24) and (25), and by using $PB_\Delta = C_\eta^{\mathrm{T}}$, the first term of (23) holds if the following inequality is satisfied

$$\begin{array}{l} \eta^{\mathrm{T}}(A_\eta^{\mathrm{T}}P + PA_\eta)\eta + w^{\mathrm{T}}B_\eta^{\mathrm{T}}P\eta + \eta^{\mathrm{T}}PB_\eta w \\ +\xi^{\mathrm{T}}\xi + \tau_F\eta^{\mathrm{T}}(M+\epsilon I)\eta - \gamma^2w^{\mathrm{T}}w + \varphi \\ \leq \eta^{\mathrm{T}}(A_\eta^{\mathrm{T}}P + PA_\eta)\eta + w^{\mathrm{T}}B_\eta^{\mathrm{T}}P\eta + \eta^{\mathrm{T}}PB_\eta w \\ +\xi^{\mathrm{T}}\xi + \tau_F\eta^{\mathrm{T}}(M+\epsilon I)\eta - \gamma^2w^{\mathrm{T}}w < 0, \end{array} \tag{26}$$

that can also be rearranged as an equivalent LMI problem in the variables $P > 0$ and $\tau_F \geq 0$ as follows

$$\begin{bmatrix} A_\eta^{\mathrm{T}}P + PA_\eta + C_\eta^{\mathrm{T}}C_\eta + \tau_F(M+\epsilon I) & PB_\eta \\ B_\eta^{\mathrm{T}}P & -\gamma^2 I \end{bmatrix} < 0, \tag{27}$$

which is the first inequality of (22) and proves the first equation of (23) and, as a consequence, completes the proof of the theorem. ∎

Remark 1. It is worth noting that there are several ways to implement the equality constraint of Theorems 1-2. One solution consists in rewriting that equality into a minimization problem and obtaining the global infimum. If this infimum equals zero, the resultant LMI variables will satisfy all the inequalities and equalities constraints [21]. Following this approach, the equality $PB_\Delta = C_\eta^{\mathrm{T}}$ of Theorems 1-2 can be replaced by the following LMI,

$$\left[\begin{array}{cc} \delta I & PB_\Delta - C_\eta^{\mathrm{T}} \\ B_\Delta^{\mathrm{T}} P - C_\eta & \delta I \end{array}\right] \geq 0,$$

where $\delta \in \mathbb{R}$ is the term to be minimized. If the problem has solution $\delta = 0$, the resultant P, τ_F, τ_J will satisfy the inequalities of Theorems 1-2 as well as the equality constraint $PB_\Delta = C_\eta^{\mathrm{T}}$. The previous minimization problem is the standard solution in the literature [21]. Another approach is to particularize by finding an special structure of the matrix P such that $PB_\Delta = C_\eta^{\mathrm{T}}$ is always guaranteed under some conditions on B_Δ and C_η. To this end, let us suppose without loss of generality that the system (1) is in observable canonical form so that $C = [0_{n-1} \;\; 1]$, and that the nonlinearity ϕ affects only the output y so that $\Delta = [0_{n-1} \;\; k_0]^{\mathrm{T}}$ with $k_0 > 0 \in \mathbb{R}$, and $B_\Delta = [0_{n-1} \;\; k_0 \;\; 0]^{\mathrm{T}}$. Under these conditions, the equality constraint $PB_\Delta = C_\eta^{\mathrm{T}}$ of Theorems 1-2 can be substituted for the following constraint over the structure of the matrix P:

$$P = P^{\mathrm{T}} = \left[\begin{array}{ccc} P_0 & 0_{n-1} & P_2 \\ 0_{n-1} & 1/k_0 & 0 \\ P_2^{\mathrm{T}} & 0 & P_5 \end{array}\right], \tag{28}$$

with $P_0 \in \mathbb{R}^{n-1 \times n-1}$, $P_2 \in \mathbb{R}^{n-1}$, $P_5 \in \mathbb{R}$. Under the previously commented conditions on B_Δ and C_η, Theorems 1-2 obtain a symmetric matrix $P > 0$ with the structure presented in (28) that satisfies $PB_\Delta = C_\eta^{\mathrm{T}}$. This can be seen, simply multiplying an arbitrary $P > 0$ with the structure of (28) by the predefined B_Δ to obtain the desired C_η.

Remark 2. Notice that $\epsilon \to 0$ as $\rho \to 0$, thus, an arbitrarily small ρ can be chosen to minimize the effect of ϵ on the stability analysis. In practice, the limit case $\rho \to 0$ is chosen so that the effect of ϵ is small enough to be neglected [12], [17].

D. Reset observer gains. Tuning and design

Analyzing the dynamics of the ReO, it is evident that there are several matrices that affect the performance of the ReO. Namely, K_P, K_I, A_ζ, and B_ζ. In [15] tuning guidelines about how to select these parameters are given. After that, we can focus on designing an appropriate parameter gain Γ. Typically, Γ is chosen to be a positive diagonal matrix in such a manner that the convergence speeds of each estimated parameter can be tuned separately. Although some authors have proposed to use time varying $\Gamma(t)$ matrix [22], [23], we consider only constant parameter gain Γ. After tuning the ReO in nominal conditions, the last step is to guarantee a bounded estimate in presence of unmodeled disturbances by choosing the threshold T_h and the leakage term σ. T_h should be chosen large enough so that $T_h > \|\theta\|$ based on prior knowledge of the system. Finally, σ is chosen to be any small positive scalar.

IV. Simulation Results

In this section, the performance of the ReO applied to an uncertain high-order nonlinear plant is shown. It can achieve a zero steady-state estimation error for all the state variables as well as for the uncertain parameter. After that, the results obtained by the ReO are compared with a PIAO with the same tuning parameters than the ReO, which is denoted by Std-PIAO, and with an optimal PIAO designed according to [6], which is denoted by J-PIAO. Notice that all these simulation results have been obtained by using Matlab-Simulink with the ode45 solver and a fixed step equal to $5 \cdot 10^{-3}[sec]$. Testing the performance of ReOs applied to real hybrid systems (e.g. bipedal robots [24]) rather than to simulation examples remains for future research.

Let us consider the following third-order noise-corrupted nonlinear system according to (1):

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_2 - 2x_3 + u + 0.2w \\ \dot{x}_2 &= -x_2 - 2x_3 - 0.5u + 0.2w \\ \dot{x}_3 &= x_2 - x_3 + 0.2(y^3 + u)\theta + 0.5u + 0.2w \\ y &= x_3 \end{aligned} \tag{29}$$

with $x(t=0) = [1.5; 0.5; 1]^{\mathrm{T}}$, $u(t) = sin(t)$, $w(t) = sin(10t)$ and an uncertain parameter $\theta = 1$.

A low-time-varying disturbance $w(t)$ is preferred rather than white noise in order to represent changes on the operating point or gradual decalibration of the system which are issues that usually arise in the adaptive observer framework [25].

Then, the aim is to develop an adaptive observer for the system described by (29) which estimates all the state variables as well as the uncertain parameter without overshooting as fast as possible. Let us begin showing the potential benefit of using a reset element in the state adaptive law. For this reason, we compare a ReO with the Std-PIAO, which is designed with the same tuning parameters than the ReO. Both observers are applied to the nonlinear system (29). Generally, PIAO for nonlinear systems are described by:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K_I z + K_P \tilde{y}, & \hat{y} &= C\hat{x}, \\ \dot{\hat{\theta}} &= \Gamma \phi^{\mathrm{T}} \tilde{y} - \Gamma \sigma_s \hat{\theta}, & \dot{z} &= A_z z + B_z \tilde{y}, \end{aligned} \tag{30}$$

where $A_z \in \mathbb{R}$ and $B_z \in \mathbb{R}$ are two tuning scalars which regulate the transient response of the integral term z, and σ_s is defined as in (3).

Fig. 1 shows how the reset element can be used to minimize the rise time without overshooting. It is evident that the ReO has a much better performance compared with the Std-PIAO, since it has a response as quick as Std-PIAO but without overshooting. The integral gain is too high for the Std-PIAO and, as consequence, it causes an oscillating estimation process. If we decrease the integral gain of the Std-PIAO to avoid overshooting it will give a slower response. However, the overshoots associated with the high integral gain are almost removed by reseting the integral term of the ReO. That result underlines the potential benefit of the reset element, due to the fact that we can decrease the settling time as long as we increase the integral gain, while we can remove the overshoots resetting the integral term. Fig. 1 also shows that Assumption 1 is satisfied since after each reset the solution is mapped to the flow set.

We also present a different tuning maximizing the performance of the observers in order to compare the ReO with an optimal PIAO designed according to [6]. Now, the ReO for the nonlinear system (29) has been tuned following the guidelines given in Section III, while the parameters of the J-PIAO are obtained by solving the minimization problem that appears in [6]. These tuning parameters as well as the state estimation error $\tilde{x}(t) = x(t) - \hat{x}(t) = [\tilde{x}_1(t); \tilde{x}_2(t); \tilde{x}_3(t)]^{\mathrm{T}}$ of both adaptive observers are shown in Fig. 2.

Comparing the results of the ReO with the J-PIAO, it can be seen that both observers achieve a fast estimation of the measured variable x_3. Nevertheless, there are significant differences in how the observers estimate the non-accessible variables x_1, x_2. As before, the ReO exploits the reset element properties to estimate x_1, x_2 as fast as the J-PIAO but without overshooting. Fig. 2 also points out that both observers could achieve zero steady-state error once they have been properly tuned.

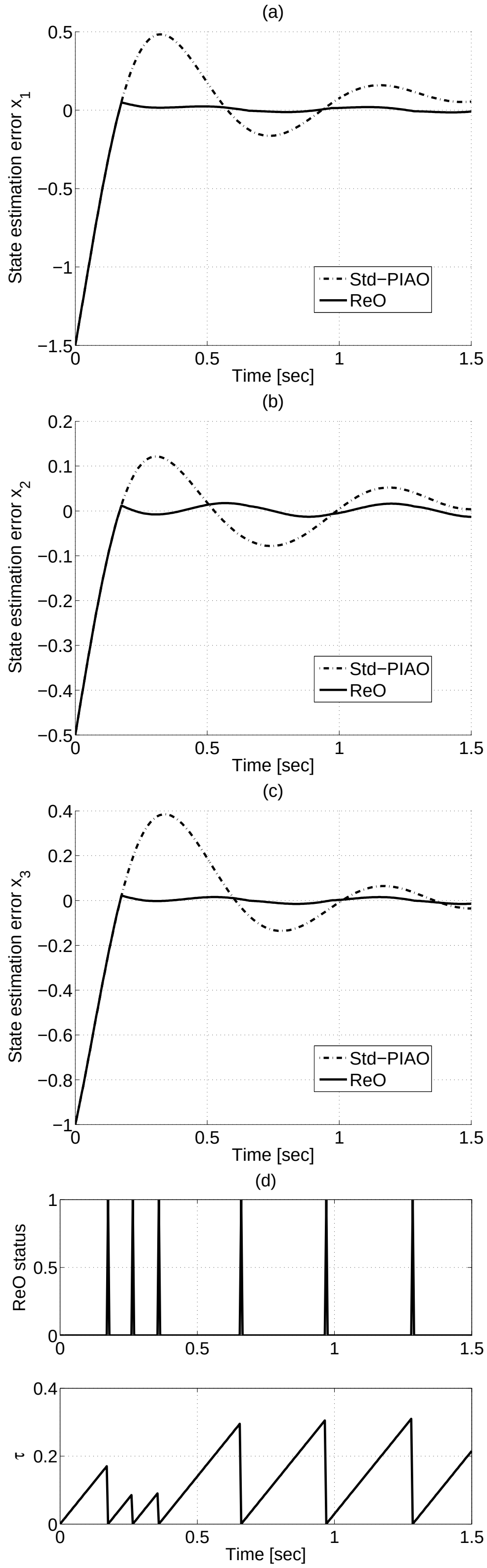

Fig. 1. Estimation results of Example 1. (a), (b), and (c) show the state estimation error $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ respectively. Dash-dot lines have been obtained by using the Std-PIAO with $\hat{x}(t=0) = [0;0;0]^{\mathrm{T}}$, $z(t=0) = 0$, $A_z = -0.1$, $B_z = 1$, $K_P = [5;1;5]^{\mathrm{T}}$, $K_I = [80;25;60]^{\mathrm{T}}$, $\Gamma = 12$, $T_h = 3.5$, and $\sigma = 10$. Solid lines have been obtained by using the ReO with $\hat{x}(t=0) = [0;0;0]^{\mathrm{T}}$, $\zeta(t=0) = 0$, $A_\zeta = -0.1$, $B_\zeta = 1$, $K_P = [5;1;5]^{\mathrm{T}}$, $K_I = [80;25;60]^{\mathrm{T}}$, $A_r = 0$, $\Gamma = 12$, $T_h = 3.5$, $\sigma = 10$, and $\rho = 10^{-2}[sec]$. Note that the system is perturbed by a periodic disturbance, being this the reason of the oscillatory behavior in the steady state. (d) shows the status of the ReO (i.e. 0: the ReO is in the flow set, 1: the ReO is in the jump set) and the temporal regularization variable τ.

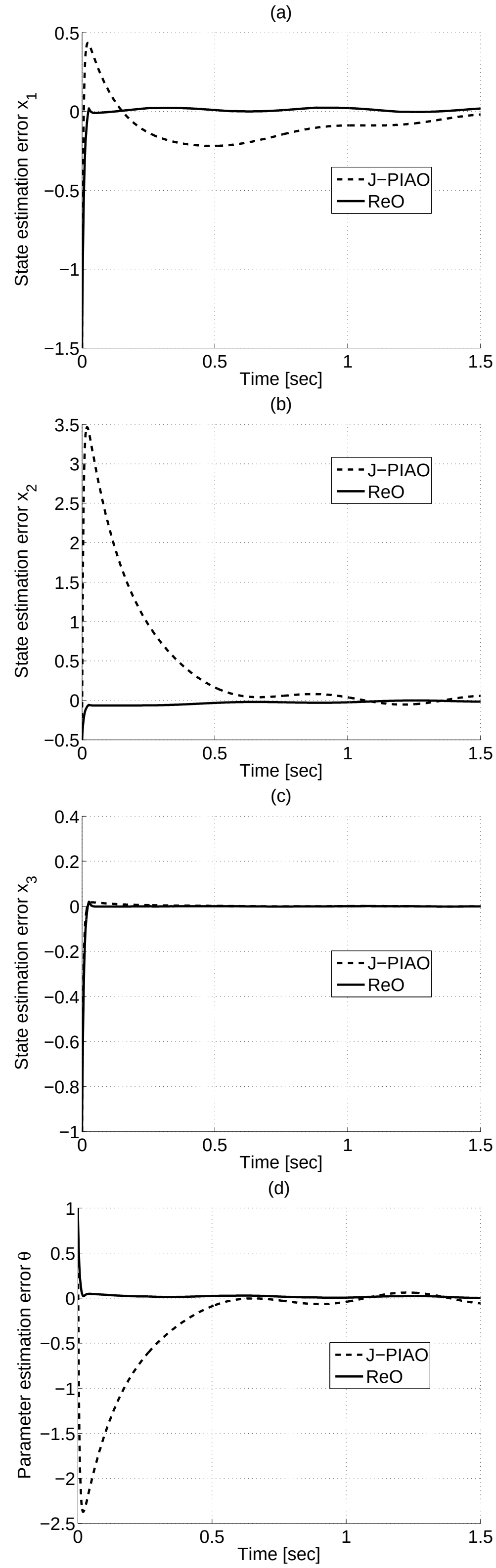

Fig. 2. Estimation results of Example 1. (a), (b), and (c) show the state estimation error $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ respectively. (d) shows the parameter estimation error $\tilde{\theta}$. Dashed lines have been obtained by using the J-PIAO with $\hat{x}(t=0) = [0;0;0]^{\mathrm{T}}$, $z(t=0) = 0$, $A_z = -0.1$, $B_z = 1$, $K_P = [400;856;200]^{\mathrm{T}}$, $K_I = [-0.0007;0.0006;1.14]^{\mathrm{T}}$, $\Gamma = 720$, $T_h = 3.5$, and $\sigma = 10$. Solid lines have been obtained by using the ReO with $\hat{x}(t=0) = [0;0;0]^{\mathrm{T}}$, $\zeta(t=0) = 0$, $A_\zeta = -0.1$, $B_\zeta = 1$, $K_P = [210;60;150]^{\mathrm{T}}$, $K_I = [2100;600;1125]^{\mathrm{T}}$, $A_r = 0$, $\Gamma = 165$, $T_h = 3.5$, $\sigma = 10$, and $\rho = 10^{-2}[sec]$.

V. Conclusion

This paper addresses the application of the ReOs to the nonlinear framework. The proposed algorithm can jointly estimate the unknown states and the uncertain parameters of a dynamic system. The stability and convergence analysis of this novel proposal has been proved by using quadratic Lyapunov functions. Moreover, a method to determine the $\mathcal{L}_2$ gain of the proposed ReO has also been developed. This method is based on a linear matrix approach which is easily computable. Simulation results have been given to highlight the potential benefit of including a reset element in the adaptive laws. Since the ReO is mainly nonlinear it can meet requirements that cannot be satisfied by pure linear observers. Namely, the reset element can decrease the overshoot and settling time of the estimation process without sacrificing the rise time. The proposed ReO is computationally simple and easy to implement in practice.

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